Recall the following properties:

1) $b^x = b^y$ if and only if $x = y$

2) $\log_b x = \log_b y$ if and only if $x = y$

3) $b^{\log_b x} = x$ $(x > 0)$

4) $\log_b b^x = x$.

1) and 2) are true because exponential b^x and $\log_b x$ are one-to one functions. so different inputs give different outputs.

Solve the following exponential equations:

a) $3^x = 81$

<u>Soln</u> $3^x = 81$

$3^x = 3^4$

Since exponenetial funtion is one-to-one, $\boxed{x = 4}$

b) $5^{7-x} = 125$

<u>Soln.</u> $5^{7-x} = 5^{-3}$

$\Rightarrow 7 - x = 3$

$\Rightarrow x = 4$

(c) $\left(\frac{1}{2}\right)^{4y} = 16$

<u>Soln.</u> $\left(\frac{1}{2}\right)^{4y} = 16$

$$\Rightarrow (2^{-1})^{4y} = 16$$
$$\Rightarrow 2^{-4y} = 2^4$$
$$\Rightarrow -4y = 4$$
$$\Rightarrow y = -1.$$

Exercise

Solve

(a) $2^{x-1} = 8$

(b) $\left(\frac{1}{3}\right)^y = 27$

Solve the following:

(i) $5^{3x} = 16$

Soln $5^{3x} = 16$

Taking natural log on both sides:

$$\ln(5^{3x}) = \ln 16$$
$$\Rightarrow 3x \ln(5) = \ln 16$$
$$\Rightarrow x = \frac{\ln 16}{3 \ln 5}$$

Note: You can take take common log. if you wish.

b) $4^{3x+2} = 71$

Soln $4^{3x+2} = 71$

Taking natural log on both sides

$\ln(4^{3x+2}) = \ln 71$

$\Rightarrow (3x+2)\ln(4) = \ln 71$

$\Rightarrow 3x+2 = \dfrac{\ln 71}{\ln 4}$

$\Rightarrow 3x = \dfrac{\ln 71}{\ln 4} - 2$

$\Rightarrow x = \dfrac{\dfrac{\ln 71}{\ln 4} - 2}{3}$

≈ 0.3583

Solve the following.

$4e^{x^2} = 64$

$\Rightarrow e^{x^2} = \dfrac{64}{4}$

$\Rightarrow e^{x^2} = 16$

$\Rightarrow \ln(e^{x^2}) = \ln(16)$

$\Rightarrow x^2 \ln(e) = \ln(16)$

$\Rightarrow x^2 \cdot 1 = \ln(16)$

$\Rightarrow x^2 = \ln 16$

$\Rightarrow x = \pm\sqrt{\ln 16}$

$\approx \pm 1.6651$

Exercise

Solve $10^{2x-3} = 7$

Solve the following $e^{2x} - 4e^{x} + 3 = 0$

Let $u = e^{x}$.

Then,

$$u^2 - 4u + 3 = 0$$

$$\Rightarrow (u-3)(u-1) = 0$$

$$\Rightarrow u = 3 \quad \text{or} \quad u = 1$$

$$\Rightarrow e^{x} = 3 \quad \text{or} \quad e^{x} = 1$$

If $e^{x} = 3$, then

$\ln e^{x} = \ln 3$

$\Rightarrow \boxed{x = \ln 3}$

If $e^{x} = 1$, then

$\ln e^{x} = \ln 1$

$\Rightarrow x = \ln 1$

$\boxed{= 0}$

Exercise

Solve $100^{x} - 10^{x} - 2 = 0.$

Logarithmic Equations

Solve $\log_4(2x-3) = \log_4(x) + \log_4(x-2)$.

Soln.

$$\log_4(2x-3) = \log_4(x(x-2))$$

Since $\log_4$ is one-to one,

$$2x-3 = x(x-2)$$

$$\Rightarrow 2x-3 = x^2-2x$$

$$\Rightarrow x^2-4x+3=0$$

$$\Rightarrow (x-3)(x-1)=0$$

$$\Rightarrow x=3 \text{ or } x=1.$$

Recall that domain of $\log_4$ is $(0,\infty)$.

If $x=1$, then $\log_4(x-2)$ is not defined.

Thus $\boxed{x=3}$.

Exercise

Solve $\ln(x+8) = \ln(x) + \ln(x+3)$.

Solve $\log_3(9x) - \log_3(x-8) = 4$

Soln

$$\log_3(9x) - \log_3(x-8) = 4$$

$$\Rightarrow \log_3\left(\frac{9x}{x-8}\right) = 4$$

$$\Rightarrow 3^4 = \frac{9x}{x-8}$$

$$\Rightarrow 81 = \frac{9x}{x-8}$$

$$\Rightarrow 9x = 81(x-8)$$

$$\Rightarrow 9x = 81x - 648$$

$$\Rightarrow x = 9$$

Exercise:

Solve $\log_2(4x) - \log_2(2) = 2.$

Word Problems

Carbon Dating:

Scientists say we humans are 200,000 years old. How did they estimate that number?

They estimate how much carbon-14 or an organism has in its body when it is alive. Then they measure how much carbon-14 is left at time of discovery. Then since carbon-14 decays at exponential rate, they can estimate the age of the fossil.

Ex.

The number of grams of carbon-14 based on radioactive decay of the isotope is given by

$$A = A_0 e^{-0.000124t}$$

where A_0 is the initial grams of carbon-14 and A is the current amount in grams.

Assume that aimals have approx. 1000 mg of carbon-14 in their bodies when they are alive. If a fossil has 200 mg of carbon 14, how old is the fossil?

Soln. We have

$$A_0 = 1000 \text{ mg}$$
$$A = 200 \text{ mg}$$
$$t = ?$$

$$200\,mg = 1000\,mg\, e^{-0.000124t}$$

$$\Rightarrow e^{-0.000124t} = 0.2$$

$$\Rightarrow -0.000124t = \ln(0.2)$$

$$\Rightarrow t = -\frac{\ln(0.2)}{0.000124}$$

$$\Rightarrow t \approx 12{,}979.338$$

— The fossil is approx 13,000 years old.

Problem:

You save \$1000 from a summer job and put it in a CD earning 5% compounding continuously. How many years will it take for your money to double?

Soln. We know

$$A = Pe^{rt}$$

Here, $P = 1000$

$r = 5\%$

$A = 2 \cdot 1000 = 2000$ (double)

$t = ?$

$$2000 = 1000\, e^{0.05t}$$

or, $2 = e^{0.05t}$

or, $\ln(2) = \ln\left(e^{0.05t}\right)$

or, $\ln(2) = 0.05t \ln(e)$

or, $t = \dfrac{\ln 2}{0.05}$

$$\approx 13.8629$$

It will take almost 14 years to double.

Problem

If \$7500 is invested in a savings account earning 5% interest compounded quaterly, how many years will pas until there is \$20,000.

Soln. We know

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Here,
$P = 7500$
$r = 5\% = 0.05$
$n = 4$ (quaterly)
$A = 20000$
$t = ?$

So,

$$20000 = 7500\left(1+\frac{0.05}{4}\right)^{4t}$$

or, $$\frac{20000}{7500} = (1.0125)^{4t}$$

or, $$\frac{8}{3} = (1.0125)^{4t}$$

Taking natural log on both sides,

$$\ln\left(\frac{8}{3}\right) = 4t\ln(1.0125)$$

or, $$t = \frac{\ln(8/3)}{4\ln(1.0125)}$$

$$\approx 19.7389$$

It will take almost 20 years.